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ON THE OPTIMAL STABILIZATION OF THE ROTATING MOTION OF A SOLID WITH A FIXED POINT BY MEANS OF FLYWHEELS

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ON THE OPTIMAL STABILIZATION OF THE ROTATING MOTION OF A SOLID WITH A FIXED POINT BY MEANS OF FLYWHEELS

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ABSTRACT. The author extends earlier results to the case of the rotation motion of a solid. The system investigated consists of a solid with a fixed point in the center of gravity. Axes of three homogeneous, symmetrical, engine-rotated flywheels lie along the principal central axes of inertia of the fixed point.

Several authors [1-3] have investigated the problem of the stabilization of the equilibrium position of a solid with a fixed point by means of flywheels (gyrostat) coupled to the solid. The author of [2] found the optimal flywheel control law which provides the extinguishing of rotating motions of the fundamental solid. The established control achieved the shift of the solid to another equilibrium configuration (from the initial one), i.e., the initial equilibrium configuration became asymptotically stable with respect to velocities, as well as with respect to the coordinates. Both solutions indicated present examples of the analytical design of control systems [4-7]. The present work is an extension of the previous investigations [3] to the case of the rotating motion of a solid. /11*

1. Statement of the Problem. Initial Motion Equations. The mechanical system under investigation (gyrostat) consists of a solid with a fixed point in the center of gravity; axes of three homogeneous symmetrical flywheels, rotated by engines, lie along the principal central axes of inertia of the fixed point. We shall introduce two systems of coordinates with the initial point in the fixed point 0: the fixed point $OX_1\ X_2\ X_3$ and the moving point $Ox_1\ x_2\ x_3$, the axes of which are directed along the axes of the flywheels. Maintaining the former designations [3], we write the equation of motion of the system in the form of three dynamic Euler-Volterra equations

$$C_1 p_1^{\cdot} + (C_3 - C_2) p_2 p_3 + p_2 H_3 - p_3 H_2 + H_1^{\cdot} = 0 \quad (123) \qquad (1.1)$$
$$(H_i = I_i \omega_i, \quad i = 1, 2, 3)$$

nine kinematic Poisson equations

$$\alpha_{i1}^{\cdot} + p_2 \alpha_{i3} - p_3 \alpha_{i2} = 0 \quad (123) \quad (i = 1, 2, 3) \qquad (1.2)$$

*Numbers given in the margin indicate pagination in original foreign text.

and three equations of the rotating motions of flywheels

$$I_i(\omega_i^{\cdot} + p_i^{\cdot}) = u_i \qquad (i = 1, 2, 3) \tag{1.3}$$

We shall restrict ourselves to the case of the symmetric gyrostat ($C_1 = C_2 = C$) and change over to new variables by introducing a new system of axes of the coordinates $y_1\ y_2\ y_3$ instead of $x_1\ x_2\ x_3$, as it is done in investigations of the rotating motions of a gyroscope [8]. The axis y_3 coincides with the axis of free rotation of a gyrostat x_3, and the axes y_1 and y_2 lie in the equatorial plane of the gyrostat x_3, and are not involved in the natural rotation ϕ. Studying the Euler-Krylov angles θ, ψ, ϕ, it is possible to direct the axis y_1 along the node line for example. The direction cosines between the axes $X_1\ X_2\ X_3$ and $y_1\ y_2\ y_3$ are designated by $\beta_{ik}(i, k = 1,2,3)$.

The change over from the variables p_i to the new variables q_i is made by the projections of the instantaneous angular velocity of the coordinate system $y_1\ y_2\ y_3$ on its axis. The vector q $\{q_1, q_2, q_3\}$ represents the angular velocity of following. We have

$$p_1 = q_1 \cos\varphi + q_2 \sin\varphi, \quad p_2 = -q_1 \sin\varphi + q_2 \cos\varphi, \quad p_3 = q_3 + \varphi^{\cdot}$$

Equations (1.1) - (1.3) are rewritten with the new variables q_i, β_{ik}. /12
The first equation in (1.1) is multiplied by cos ϕ, and the second by -sin ϕ and they are added, then the first equation is multiplied by sin ϕ, and the second by cos ϕ and they are again added; the third equation in (1.1) is retained in its original form. After simple transformations, we obtain the following system of equations:

$$\begin{gathered} Cq_1^{\cdot} + (C_3 - C)q_2q_3 + C_3\varphi^{\cdot}q_2 + q_2G_3 - q_3G_2 + G_1^{\cdot} = 0 \\ Cq_2^{\cdot} + (C - C_3)q_1q_3 - C_3\varphi^{\cdot}q_1 + q_3G_1 - q_1G_3 + G_2^{\cdot} = 0 \\ C_3(q_3 + \varphi^{\cdot})^{\cdot} + q_1G_2 - q_2G_1 + G_3^{\cdot} = 0 \end{gathered} \tag{1.4}$$

where

$$G_1 = H_1 \cos\varphi - H_2 \sin\varphi, \quad G_2 = H_1 \sin\varphi + H_2 \cos\varphi, \quad G_3 = H_3 \tag{1.5}$$

Performing analogous transformations with the equations (1.3) with the assumption that $I_1 = I_2 = I$, we obtain the system of equations

$$\begin{gathered} G_1^{\cdot} + Iq_1^{\cdot} + (G_2 + Iq_2)\varphi^{\cdot} = w_1, \quad G_2^{\cdot} + I\,q_2^{\cdot} - (G_1 + Iq_1)\,\varphi^{\cdot} = w_2 \\ G_3^{\cdot} + I_3(q_3 + \varphi^{\cdot})^{\cdot} = w_3 \end{gathered} \tag{1.6}$$

where

$$w_1 = u_1 \cos\varphi - u_2 \sin\varphi, \quad w_2 = u_1 \sin\varphi + u_2 \cos\varphi, \quad w_3 = u_3 \tag{1.7}$$

The equation system (1.2) takes the form

$$\beta_{i1}^{\cdot} + q_2\beta_{i3} - q_3\beta_{i2} = 0 \quad (i = 1, 2, 3) \quad (123) \tag{1.8}$$

The equations (1.4) and (1.8) permit a particular solution corresponding to the uniform rotation of a gyrostat with an angular velocity ω around a fixed axis X_3

$$\varphi^{\cdot} = \omega, \quad q_i = 0, \quad \beta_{ik} = \begin{cases} 1, & i = k \\ 0, & i \neq k \end{cases} \quad G_1 = G_2 = 0, \quad G_3 = G_3^0$$
$$w_i = 0 \qquad (i, k = 1, 2, 3) \tag{1.9}$$

In order to exclude the kinetic moments G_i from the equations (1.6), use is made of the law of the conservation of the moment of momentum of a gyrostat in the projections on the axis X_1 X_2 X_3

$$(Cq_1 + G_1)\beta_{i1} + (Cq_2 + G_2)\beta_{i2} + [C_3(q_3 + \varphi^{\cdot}) + G_3]\beta_{i3} = h_i = \text{const} \quad (i = 1, 2, 3) \tag{1.10}$$

which corresponds to the three integral equations (1.5).

The quantity G_i is determined from Equation (1.10) and substituted in Equation (1.6), taking Equation (1.8) into consideration. Then, after transformations, we obtain the system of three equations which does not contain G_i

$$\begin{aligned} (C - I)\, q_1^{\cdot} &= -(C - I)\, q_2\varphi^{\cdot} + (q_3 + \varphi^{\cdot}) \sum_i h_i\beta_{i2} - q_2 \sum_i h_i\beta_{i3} - w_1 \\ (C - I)\, q_2^{\cdot} &= -(C - I)\, q_1\varphi^{\cdot} - (q_2 + \varphi^{\cdot}) \sum_i h_i\beta_{i1} + q_1 \sum_i h_i\beta_{i3} - w_2 \\ (C_3 - I_3)(q_3 + \varphi^{\cdot})^{\cdot} &= q_2 \sum_i h_i\beta_{i1} - q_1 \sum_i h_i\beta_{i2} - w_3 \end{aligned} \tag{1.11}$$

Here and later the summation is performed from 1 to 3 according to the corresponding subscript.

For the investigated stationary motion (1.9) the constants are equal to

$$h_1^{\circ} = h_2^{\circ} = 0, \qquad h_3^{\circ} = C_3\omega + G_3^{\circ}$$

Thus, the gyrostat motion is completely described by the twelve equations in (1.8) and (1.11). The phase coordinates of the system are q_i, β_{ik} (i, k = = 1,2,3); however, only six of them are independent, since the direction cosines β_{ik} are coupled by six geometrical relations /13

$$\sum_i \beta_{ki}\beta_{li} = \begin{cases} 1, & k = l \\ 0, & k \neq l \end{cases} \quad (k, l = 1, 2, 3) \tag{1.12}$$

which may be viewed as integrals of the equations (1.8).

2. Solution to the Stabilization Problem. In order to study the stability (1.9) with respect to q_1, q_2, $q_3 + \phi$, β_{ik} (i, k = 1.2.3) an equation of perturbed motion is composed, retaining for the perturbations the designations of the initial variables. The constants h_i will designate the initial perturbations. In this case, use is made of the "shortened" system of perturbed motion equations [3], which is the totality of the complete equations (1.8) and the system of the first approximation (1.11), since from the asymptotic stability of motion (1.9), in virtue of the "shortened" system of equations, ensues the asymptotic stability (1.9), in virtue of the complete system of equations (1.8) and (1.11). The "shortened" system of equations has the form

$$
\begin{aligned}
q_1^{\cdot} &= h_{12}q_3 - (h_{13} + \omega)\, q_2 + \omega \sum_i h_{1i}\beta_{i2} + v_1 \\
q_2^{\cdot} &= (h_{13} + \omega)\, q_1 - h_{11}q_3 - \omega \sum_i h_{1i}\beta_{i1} + v_2 \\
q_3^{\cdot} &= h_{31}q_2 - h_{32}q_1 + v_3
\end{aligned}
\tag{2.1}
$$

$$
\frac{d\beta_{ii}}{dt} = B_{ii} \quad (i = 1,\ 2,\ 3), \qquad \frac{d\beta_{12}}{dt} = -q_3 + B_{12}, \qquad \frac{d\beta_{13}}{dt} = q_2 + B_{13} \quad (123)
$$

где

$$
\frac{h_i}{C-I} = h_{1i}, \quad \frac{h_i}{C_3 - I_3} = h_{3i} \quad (i = 1,\ 2), \quad \frac{h_3^0 + h_3}{C-I} = h_{13} \tag{2.2}
$$

$$
(C - I)\, v_1 = -w_1 + \omega h_2, \quad (C - I)\, v_2 = -w_2 - \omega h_1, \quad (C_3 - I_3)\, v_3 = -w_3
$$

$$
B_{i1} = q_3\beta_{i2} - q_2\beta_{i3} \quad (i = 1,\ 2,\ 3) \quad (123) \tag{2.3}
$$

Here the problem consists in the following: to determine the functions v_i of phase coordinates so that the zeroth solution

$$
q_i = 0, \quad \beta_{ik} = 0 \quad (i,\ k = 1,\ 2,\ 3) \tag{2.4}
$$

will be asymptotically stable and, furthermore, that the condition of the minimum of the integral is satisfied

$$
\int_0^\infty \Omega(q_1,\ q_2,\ q_3,\ \beta_{11}, \ldots, \beta_{33},\ v_1,\ v_2,\ v_3)\, dt
$$

where Ω is a certain positive function which will be found in the course of the solution of the problem. A priori only the structure of Ω is defined; it is assumed that

$$
\begin{aligned}
\Omega = {} & F_1(q_1,\ q_2,\ q_3) + F_2(\beta_{11}, \ldots, \beta_{33}) + \sum_i n_i v_i^2 + \\
& + \Lambda(q_1,\ q_2,\ q_3,\ \beta_{11}, \ldots, \beta_{33})
\end{aligned}
\tag{2.5}
$$

here

$$
F_1(q_1,\ q_2,\ q_3) = \sum_{i,k} e_{ik} q_i q_k
$$

It should be borne in mind that corresponding restrictions will be imposed on the coefficients e_{ik} ($e_{ii} > 0$), $n_i > 0$ henceforth; the function F_2 is to be determined, Λ denotes the possible terms of an order higher than the second. /14
The function F_1 should be the positive definite quadratic form of a positive definite quadratic form β_{ik} (i, k = 1,2,3).

In order to solve the problem posed of the analytic design of control systems, the same as was done above, use is made of the fundamental theorem of the second method Lyapunov used in the investigation of optimal stabilization problems [9]. According to the theorem, the optimal control $v_i°$ and the optimal Lyapunov function V° satisfy the following system of four equations in partial derivatives of the first order:

$$\sum_i \frac{\partial V^\circ}{\partial q_i}(Q_i + v_i^\circ) + \sum_i \delta_i q_i + \sum_{i,k}\frac{\partial V^\circ}{\partial \beta_{ik}} B_{ik} + \\ + \Omega(q_1, q_2, q_3, \beta_{11}, \ldots, \beta_{33}, v_1^\circ, v_2^\circ, v_3^\circ) = 0 \tag{2.6}$$

$$\frac{\partial V^\circ}{\partial q_i} + 2n_i v_i^\circ = 0 \quad (i = 1, 2, 3)$$

Here

$$Q_1 = h_{12}q_3 - (h_{13} + \omega) q_2 + \omega \sum_i h_{1i}\beta_{i2}, \quad Q_2 = (h_{13} + \omega) q_1 - h_{11}q_3 - \\ - \omega \sum_i h_{1i}\beta_{i1}, \qquad Q_3 = h_{31}q_2 - h_{32}q_1, \qquad \delta_1 = \frac{\partial V^\circ}{\partial \beta_{32}} - \frac{\partial V^\circ}{\partial \beta_{23}} \quad (123) \tag{2.7}$$

Since

$$v_i^\circ = -\frac{1}{2n_i}\frac{\partial V^\circ}{\partial q_i} \quad (i = 1, 2, 3) \tag{2.8}$$

then for the function V° one nonlinear equation is obtained in partial derivatives of the first order

$$-\sum_i \frac{1}{4n_i}\left(\frac{\partial V^\circ}{\partial q_i}\right)^2 + \sum_i \frac{\partial V^\circ}{\partial q_i} Q_i + \sum_i \delta_i q_i + \sum_{i,k} \frac{\partial V^\circ}{\partial \beta_{ik}} B_{ik} + \\ + F_1(q_1, q_2, q_3) + F_2(\beta_{11}, \ldots, \beta_{33}) + \Lambda(q_1, q_2, q_3, \beta_{11}, \ldots, \beta_{33}) = 0 \tag{2.9}$$

On the basis of Equation (1.12) the variables β_{ik} are coupled by six relationships (the integrals of Equation (2.2))

$$\Phi_{kl} = \beta_{kl} + \beta_{lk} + \sum_i \beta_{ki}\beta_{li} = 0 \qquad (k, l = 1, 2, 3;\ k \leqslant l) \tag{2.10}$$

The Lyapunov function is sought in the form of a quadratic configuration with undetermined coefficients [3]

$$2V^\circ = 2\Phi_0 + \sum_i k_i \Phi_{ii}$$

Here

$$2\Phi_0 = -2\sum_i k_i\beta_{ii} + \sum_i m_i q_i^2 + 2q_1\sum_{i,k} a_{ik}\beta_{ik} + 2q_2\sum_{i,k} b_{ik}\beta_{ik} + $$
$$+ 2q_3\sum_{i,k} c_{ik}\beta_{ik} \quad (k_i > 0,\ m_i > 0) \tag{2.11}$$

Consequently,

$$\frac{\partial V^\circ}{\partial q_1} = m_1 q_1 + \sum_{i,k} a_{ik}\beta_{ik}, \quad \frac{\partial V^\circ}{\partial q_2} = m_2 q_2 + \sum_{i,k} b_{ik}\beta_{ik}, \quad \frac{\partial V^\circ}{\partial q_3} = m_3 q_3 + \sum_{i,k} c_{ik}\beta_{ik} \tag{2.12}$$

Substituting Equation (2.11) in Equation (2.9), an algebraic equation system /15
is obtained which couples the coefficients of the functions V° and Ω

$$\begin{gathered} d_1^2 n_1 + a_{23} - a_{32} = e_{11}, \quad d_2^2 n_2 + b_{31} - b_{13} = e_{22}, \quad d_3^2 n_3 + c_{12} - c_{21} = e_{33} \\ (h_{13} + \omega)(m_1 - m_2) - a_{13} + a_{31} - b_{32} + b_{23} = 2e_{12} \\ -h_{12} m_1 + h_{32} m_3 - a_{21} + a_{12} - c_{32} + c_{23} = 2e_{13} \\ h_{11} m_2 - h_{31} m_3 - b_{21} + b_{12} - c_{13} + c_{31} = 2e_{23} \\ (d_i = m_i/2n_i;\ i = 1,\ 2,\ 3) \end{gathered} \tag{2.13}$$

The remaining equations are divided into nine subsystems, linear with respect to a_{ik}, b_{ik}, c_{ik}, each of which contains three coefficients corresponding to the identical subscripts i, k = 1,2,3. All the subsystems have the same determinant

$$\Delta = \begin{vmatrix} -d_1 & h_{13} + \omega & -h_{32} \\ -(h_{13} + \omega) & -d_2 & h_{31} \\ h_{12} & -h_{11} & -d_3 \end{vmatrix} \tag{2.14}$$

and the right hand terms include components containing ω and the parameters k_i and m_i.

With sufficiently great d_i the determinant Δ is deliberately not equal to zero and each subsystem has a single solution for a_{ik}, b_{ik}, c_{ik}, in the form of the functions h_{ik} and ω. The following designations are introduced:

$$\begin{gathered} d_2 d_3 + h_{11} h_{31} = \lambda_{11}, \quad -d_3(h_{13} + \omega) - h_{11} h_{32} = \lambda_{12} \\ -d_2 h_{32} + h_{31}(h_{13} + \omega) = \lambda_{13} \\ d_3(h_{13} + \omega) - h_{12} h_{31} = \lambda_{21}, \quad d_1 d_3 + h_{12} h_{32} = \lambda_{22} \\ -d_1 h_{31} - h_{32}(h_{13} + \omega) = \lambda_{23} \\ d_2 h_{12} + h_{11}(h_{13} + \omega) = \lambda_{31}, \quad d_1 h_{11} - h_{12}(h_{13} + \omega) = \lambda_{32} \\ d_1 d_2 + (h_{13} + \omega)^2 = \lambda_{33} \\ 1 / \Delta = \mu \end{gathered}$$

Then, the indicated solutions will be written as follows:

$$
\begin{aligned}
&a_{11} = \mu m_2 h_{11}\omega\lambda_{12}, \quad b_{11} = \mu m_2 h_{11}\omega\lambda_{22}, \quad c_{11} = -\mu m_2 h_{11}\omega\lambda_{32}\\
&a_{12} = \mu\,(k_1\lambda_{13} - m_1 h_{11}\omega\lambda_{11}) \quad a_{21} = -\mu\,(k_2\lambda_{13} + m_2 h_{12}\omega\lambda_{12})\\
&b_{12} = \mu\,(-k_1\lambda_{23} + m_1 h_{11}\omega\lambda_{21}), \quad b_{21} = \mu\,(k_2\lambda_{23} + m_2 h_{12}\omega\lambda_{22})\\
&c_{12} = \mu\,(k_1\lambda_{33} - m_1 h_{11}\omega\lambda_{31}), \quad c_{21} = -\mu\,(k_2\lambda_{33} + m_2 h_{12}\omega\lambda_{32})\\
&a_{13} = \mu k_1\lambda_{12}, \quad b_{13} = -\mu k_1\lambda_{22}, \quad c_{13} = \mu k_1\lambda_{32}\\
&a_{22} = -\mu m_1 h_{12}\omega\lambda_{11}, \quad b_{22} = \mu m_1 h_{12}\omega\lambda_{21}, \quad c_{22} = -\mu m_1 h_{12}\omega\lambda_{31}\\
&a_{23} = \mu k_2\lambda_{11}, \quad b_{23} = -\mu k_2\lambda_{21}, \quad c_{23} = \mu k_2\lambda_{31}\\
&a_{31} = -\mu\,(k_3 + m_2 h_{13}\omega)\lambda_{12}, \quad a_{32} = -\mu\,(k_3 + m_1 h_{13}\omega)\lambda_{11}\\
&b_{31} = \mu\,(k_3 + m_2 h_{13}\omega)\lambda_{22}, \quad b_{32} = \mu\,(k_3 + m_1 h_{13}\omega)\,\lambda_{21}\\
&c_{31} = -\mu\,(k_3 + m_2 h_{13}\omega)\lambda_{32}, \quad c_{32} = -\mu\,(k_3 + m_1 h_{13}\omega)\lambda_{31}\\
&a_{33} = b_{33} = c_{33} = 0
\end{aligned}
\tag{2.15}
$$

It follows from the expressions (2.13) - (2.15) that at sufficiently large d_i the functions V° and F_1 are positive definite. The function F_2 consists of the fundamental function $F_2{}^*$ and the supplementary function $F_2{}^{**}$ of the quadratic forms

$$
\begin{aligned}
F_2{}^* &= \frac{1}{4n_1}\Big(\sum_{i,k} a_{ik}\beta_{ik}\Big)^2 + \frac{1}{4n_2}\Big(\sum_{i,k} b_{ik}\beta_{ik}\Big)^2 + \frac{1}{4n_3}\Big(\sum_{i,k} c_{ik}\beta_{ik}\Big)^2\\
F_2{}^{**} &= \omega\Big(\sum_i h_{1i}\beta_{i1}\Big)\Big(\sum_{i,k} b_{ik}\beta_{ik}\Big) - \omega\Big(\sum_i h_{1i}\beta_{i2}\Big)\Big(\sum_{i,k} a_{ik}\beta_{ik}\Big)
\end{aligned}
\tag{2.16}
$$

Since the nine variables of β_k are coupled by the six relationships (2.10) /16 the form of the function $F_2{}^*$ with sufficiently large d_i may be made positive definite [3].

The form $F_2{}^{**}$ is alternating. Therefore, in order for the function

$$F_2 = F_2{}^* + F_2{}^{**}$$

to be positive definite, it is necessary to introduce an upper boundary on the coefficients of the form $F_2{}^{**}$ in a corresponding manner. This will lead to the inequalities which limit the initial moment of momentum of the gyrostat, $h_3{}^\circ$, and the initial perturbations, h_i. The inequalities indicated will be established below.

The terms greater than the second order Λ should be taken in the form

$$\Lambda\,(q_1, q_2, q_3, \beta_{11}, \ldots, \beta_{33}) = -\sum_{i,k}(q_1 a_{ik} + q_2 b_{ik} + q_3 c_{ik})\,B_{ik} \tag{2.17}$$

Thus, it has been established that the motion (2.4) is stabilized by means of control (2.8), (2.12), and (2.15), if: 1) the forms of $V°$ and F_2 are positive definite; 2) the inequality $\Delta \neq 0$ is valid; 3) the initial moment of momentum $h_3°$, and the region of the initial perturbations, h_i, are selected from the fixed-sign configurations of the form F_2. The lower boundary of $m_i/2n_i$ can be calculated from these conditions with the fixed parameters of k_i and m_i and given h_{ik} numbers.

In this case, the control found proves to be optimal in the sense of the minimum of the integral of the function Ω (2.5), (2.13) - (2.17).

Returning to former designations, according to (1.7) and (2.3), the authors obtain the following analytical expressions of the control moments of the engines, providing for the optimal stabilization of the stationary motion of a gyrostat (1.9):

$$
\begin{aligned}
u_1°(t, q, \beta) &= \omega(h_2 \cos\omega t - h_1 \sin\omega t) + (C - I)\Big[d_1 q_1 \cos\omega t + d_2 q_2 \sin\omega t + \\
&+ \frac{1}{2n1}\Big(\sum_{i,k} a_{ik}\beta_{ik}\Big)\cos\omega t + \frac{1}{2n_2}\Big(\sum_{i,k} b_{ik}\beta_{ik}\Big)\sin\omega t\Big] \\
u_2°(t, q, \beta) &= -\omega(h_2 \sin\omega t + h_1 \cos\omega t) + (C - I)\Big[-d_1 q_1 \sin\omega t + \\
&+ d_2 q_2 \cos\omega t - \frac{1}{2n_1}\Big(\sum_{i,k} a_{ik}\beta_{ik}\Big)\sin\omega t + \frac{1}{2n_2}\Big(\sum_{i,k} b_{ik}\beta_{ik}\Big)\cos\omega t\Big] \\
u_3°(t, q, \beta) &= (C_3 - I_3)\Big(d_3 q_3 + \frac{1}{2n_3}\sum_{i,k} c_{ik}\beta_{ik}\Big)
\end{aligned}
\tag{2.18}
$$

The optimal control (2.18) is externally linear with respect to the perturbations of the velocities, q_i, the perturbations of the coordinates β_{ik}, and the initial perturbations, h_i, with periodic coefficients. However, not all the terms included in the functions (2.18) have the same order of smallness, since the coefficients a_{ik}, b_{ik}, and c_{ik}, according to (2.15), are composed of terms, some of which are finite quantities, and others have first, second, and third orders of smallness according to the initial perturbations of h_i. Thus, if q_i, β_{ik} h_i are considered as quantities of one order, the control found will contain terms of the first, second, third, and fourth orders of smallness, i.e., they are actually nonlinear.

The expressions (2.18) indicate that in order to realize the control found it is necessary to have devices which measure the initial perturbations, h_i, and the phase coordinates of the solid, q_i, β_{ik}. Since in real conditions the initial perturbations are of a random nature, it is expedient to present the solution to the problem in the probabilistic formulation, viewing the initial perturbations as certain random quantities with unknown probabilistic characteristics.

3. Shift to Independent Variables. Let us change over from the dependent variables β_{ik} to the independent variables, i.e., the Krylov angles θ and ψ plotting the directing cosines $\phi = 0$ in the table [10]. Then the perturbation β_{ik} is expressed by means of the perturbation of the angles θ and ψ as follows: /17

$$\beta_{13} = -\beta_{31} = \psi + \ldots, \qquad \beta_{32} = -\beta_{23} = 0 + \ldots, \ \beta_{21} = 0 \tag{3.1}$$

The remaining β_{ik} start with the terms of the second order of smallness. The function F_2 will take the form

$$F_2(0, \psi) = A_1\psi^2 + 2A_{12}0\psi + A_2 0^2 + \frac{1}{4n_3}[(c_{13} - c_{31})\psi + (c_{32} - c_{23})0]^2 + \ldots$$

where

$$\begin{aligned} A_1 &= \frac{1}{4n_1}(a_{13} - a_{31})^2 + \frac{1}{4n_2}(b_{13} - b_{31})^2 - h_{13}\omega(b_{13} - b_{31}) \\ A_2 &= \frac{1}{4n_1}(a_{32} - a_{23})^2 + \frac{1}{4n_2}(b_{32} - b_{23})^2 - h_{13}\omega(a_{32} - a_{23}) \\ A_{12} &= \frac{1}{4n_1}(a_{13} - a_{31})(a_{32} - a_{23}) + \frac{1}{4n_2}(b_{13} - b_{31})(b_{32} - b_{23}) - \\ &\quad - \frac{1}{2}h_{13}\omega(b_{32} - b_{23} + a_{13} - a_{31}) \end{aligned} \tag{3.2}$$

The fixed-sign configurations of the form (3.2) are expressed by the inequalities

$$A_1 > 0, \qquad A_1A_2 - A_{12}^2 > 0 \tag{3.3}$$

The inequalities (3.3), according to (2.15), establish the coupling between the initial moment of momentum, $h_3°$, and the initial perturbation, h_i, and by the selected parameters k_i, m_i, and n_i.

We shall consider h_i as small as desired and find the limits on $h_3°$. It is assumed that

$$m_i = m, \quad k_i = k, \quad n_i = n, \quad d_i = d \quad (i = 1, 2, 3)$$

Omitting the terms containing h_{11}, h_{12}, h_{31}, and h_{32} in Equation (2.15), we obtain

$$a_{13} = b_{23}, \quad a_{31} = b_{32}, \quad a_{23} = -b_{13}, \quad a_{32} = -b_{31}$$

This, on the basis of Equation (3.3), leads to

$$A_1 = A_2 > 0, \qquad A_{12} = 0 \tag{3.4}$$

After computations, we obtain

$$\frac{h_3^\circ \omega}{C - I} < \frac{2k}{m} \tag{3.5}$$

It is assumed that all three flywheels do not rotate relative to the solid being subjected to stabilization in the stationary motion being studied, i.e.,

$$G_3^\circ = 0, \qquad h_3^\circ = C_3\omega \tag{3.6}$$

Then from equation (3.5) it follows that

$$\omega < \left(\frac{2k}{m}\frac{C-I}{C_3}\right)^{1/2}$$

Coefficients of the form V° and F_1, as can be easily seen, have the property that the fixed-sign configurations of the forms are not disrupted with an increase of k if in this case a corresponding increase of d is achieved (by means of decreasing n). Thus, the found control ensures the stabilization of the rotating motion of a solid for a wide (theoretically as large as desired) range of angular velocities, ω; furthermore, this is easier to achieve the greater $C - I$ as compared to C_3, i.e., the greater the extension of the solid along the axis of rotation. In practice, the selection of d is limited by the capacity of the engine, hence a specific upper boundary is obtained for ω. It may be seen from Equation (3.5) that the range of permissible ω may be expanded by means of decreasing h_3° selecting G_3° not equal to zero, and having a sign opposite to $C_3\omega$.

The minimized functional assumes in the independent variables a simple form /18

$$\int_0^\infty [b_1(q_1^2 + q_2^2) + b_2 q_3^2 + a(\psi^2 + \theta^2) + n(v_1^2 + v_2^2 + v_3^2) + \ldots]\, dt$$

Here

$$b_1 = d^2 n - \frac{d(2k + ml\omega^2)}{d^2 + (1 + l^2)\omega^2}, \qquad b_2 = d^2 n - \frac{2k}{d}$$

$$a = \frac{4k^2 - m^2 l^2 \omega^4}{4n[d^2 + (1 + l)^2\omega^2]} \qquad \left(l = \frac{C_3}{C - I}\right)$$

The dotted lines designate the terms of a higher order of smallness (taking into account the smallness of the initial perturbations, h_i).

By means of the scheme proposed for stabilization, it may be seen from the formulas presented that the easier the stabilization of the rotation of the solid, the smaller its angular velocity. At the same time it is known that the greater the initial moment of momentum of the gyrostat [11] (in particular, a gyroscope), the more stable the latter in the sense of Lyapunov. However, a fast rotating gyroscope, being more stable in the usual sense than a slower rotating gyroscope, is "lazier": it is difficult to displace the faster gyroscope from the initial position. urthermore, it is more difficult to return it to its original position. In other words a faster rotating gyroscope is more difficult to make asymptotically stable than a slower rotating gyroscope. This simple mechanical fact provides the explanation for the results obtained.

It may also be established that with increasing ω, the region of permissible initial perturbations, h_i, decreases (and the opposite is also true).

Rotating space objects in some cases become unstable [12]. The results obtained may prove to be useful in an investigation of techniques for stabilizing these objects.

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